ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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ABSTRACT

Let A be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $U = \{ z \in C : |z| < 1 \}$ of the complex plane. In this paper we introduce a subclass $J(\lambda, \alpha, \delta)$ of A and study some of their interesting properties such as inclusion results and covering theorem.

AMS Subject Classification: 30C45.

Keywords: Analytic functions, univalent functions with positive real part, convex functions, convolution, integral operator.

I. INTRODUCTION

Let A be the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the open disc $U = \{ z \in C : |z| < 1 \}$. Let S denote the class of functions in A which are univalent in U.

Let $P_k(\delta)$ be the class of functions $p(z)$ analytic in U satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \delta}{1 - \delta} \right| d\theta \le k\pi \tag{1.1}$$

where $z = re^{i\theta}, k \ge 2$ and $\theta \le \delta < 1$. This class has been introduced in [5]. We note that, for $\delta = 0$, we obtain the class P_k defined and studied in [6] and for $\delta = 0, k = 2$ we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\delta)$ of functions with positive real part greater than δ.

From (1.1) we can easily deduce that $p \in P_k(\delta)$ if and only if, there exists $p_1, p_2 \in P(\delta)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2(z)$$

For the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ which are analytic in U, let $(f * g)(z)$ denote the convolution of $f(z)$ and $g(z)$ and be defined by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$$

Now we consider the incomplete beta function $\phi(a, c; z)$ which is defined by

$$\phi(a, c; z) = 2F(1, a, c; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \; c \ne 0, -1, -2 \ldots$$

corresponding to the function $\phi(a, c; z)$ Carlson and Shaffer [1] defined a linear operator $L(a, c)$ and A defined by

$$L(a, c)(f(z)) = \phi(a, c; z) \times f(z), \; f \in A$$

It is known in [1] that $L(a, c)$ maps A into itself. If $a \ne 0, -1, -2, \ldots$ then $L, a, c)$ has a continuous inverse of $L(a, c)$. If $c > a > 0$, then $L(a, c)$ has the integral representation.

$$L(a, c)(f(z)) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}\frac{f(tz)}{t}\,dt$$

Definition 1.1 Let $f \in A$, then $f \in J(\lambda, \alpha, \delta)$ if and only if

$$\left\{(1-\lambda)\left[\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\right] + \lambda\,[z\alpha f''(z) + f'(z)]\right\} \in P_k(\delta),\ z \in U$$

where $a > 0, \lambda > 0, k \geq 2$ and $0 \leq \delta < 1$

Lemma 1.1 If $p(z)$ is analytic in U with $p(0) = 1$ and λ is a complex number satisfying $\operatorname{Re}\lambda > 0\ (\lambda \neq 0)$ then

$$\operatorname{Re}[p(z) + \lambda zp'(z)] > \beta\ (0 \leq \beta < 1)$$

implies.

$$\operatorname{Re} p(z) > \beta + (1-\beta)(2\gamma - 1)$$

where γ is given by

$$\gamma = \int_0^1 (1 + t^{\operatorname{Re}\lambda}) - 1\,dt$$

Lemma 1.2 [4] Let $c > 0, \lambda > 0, \delta < 1$. If $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be analytic in U and

$$\operatorname{Re}[p(z) + c\lambda zp'(z)] > \delta,\ z \in U$$

then

$$\operatorname{Re}[p(z) + czp'(z)] \geq 2\delta - 1 + \frac{1-\delta}{\lambda} + 2(1-\delta)\left(1 - \frac{1}{\lambda}\right)\frac{1}{c\lambda}\int_0^1 \frac{t^{\frac{1}{c\lambda}} - 1}{1+t}\,dt$$

This result is sharp.

II. MAIN RESULTS

Theorem 2.1 Let $\lambda, a > 0, 0 \leq \delta < 1$ and let $f \in J(\lambda, \alpha, \delta)$. Then $\alpha f'(z) + (1-\alpha)\frac{f(z)}{z} \in P_k(\delta_1)$ where δ_1 is given by

$$\delta_1 = \delta + (1-\delta)(2\gamma - 1) \tag{2.1}$$

and

$$\gamma = \int_0^1 (1 + t^{\operatorname{Re}\gamma})^{-1}\,dt$$

Proof. Let

$$\alpha f'(z) + (1-\alpha)\frac{f(z)}{z} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z)$$

Then $p(z)$ is analytic in U and

$$p(z) = \alpha f'(z) + (1-\alpha)\frac{f(z)}{z} \tag{2.2}$$

Taking the derivatives on both sides we get,

$$(1-\lambda)\left[\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\right] + \lambda\,[z\alpha f''(z) + f'(z)] = p(z) + \lambda zp'(z).$$

Since $f \in J(\lambda, \alpha\,\delta)$, $p(z) + \lambda zp'(z) \in P_k(\delta), z \in U$.

This implies that $\operatorname{Re}[p(z) + \lambda zp'(z)] > \delta$. Using Lemma 1.1, $\operatorname{Re}[p(z)] > \delta_1$ where δ_1 is given by (2.1).

Hence $p(z) \in P_k(\delta_1), z \in U$

This completes the proof.

Theorem 2.2 Let $\alpha > 0, \lambda > 0, 0 \leq \delta < 1$. If $f \in J(\lambda, \alpha, \delta)$ then $z\alpha f''(z) + f' \in P_k(\delta_2)$ where

$$\delta_2 = 2\delta - 1 + \frac{1-\delta}{\lambda} + 2(1-\delta)\left(1 - \frac{1}{\delta}\right)\frac{1}{\lambda}\int_0^1 \frac{t^{\frac{1}{\lambda}} - 1}{1+t}\,dt$$

This result is sharp.

Proof Ler $p(z) = \alpha f'(z) + (1-\alpha)\dfrac{f(z)}{z}$

Taking derivatives on both sides, we get

$$(1-\lambda)\, p(z) + \lambda\,[z\alpha f''(z) + f'(z)] = p(z) + \lambda\, zp'(z)$$

Since $f \in J(\lambda, \alpha\,\delta)$ we have

$\text{Re}\,[p(z) + \lambda\, zp'(z)] > \delta,\ z \in U$

According to Lemma 1.2

$\text{Re}\,[z\,\alpha f''(z) + f'(z)] = \text{Re}\,[p(z) + \lambda\, zp'(z)]$

$$\geq 2\delta - 1 + \frac{1-\delta}{\lambda} + 2(1-\delta)\left(1 - \frac{1}{\lambda}\right)\frac{1}{\lambda}\int_0^1 \frac{t^{\frac{1}{\lambda}-1}}{1+t}\,dt$$

But

$$f_{\lambda,\alpha\,\delta}(z) = z\left[\frac{1}{\lambda}\int_0^1 t^{\frac{1}{\lambda}-1}\,\frac{1+(1-2\delta)\,tz}{1-tz}\,dt\right]^{\frac{1}{a}} \in J(\lambda, \alpha, \delta)$$

Hence the inequality is sharp.

Theorem 2.3 For each $\alpha > 0,\ 0 \leq \lambda_1 < \lambda_2$,

$J(\lambda_2, \alpha, \delta) \subset J(\lambda_1, \alpha, \delta)$

Proof. For $\lambda_1 = 0$, the proof is immediate

Let $\lambda_1 > 0$ and let $f \in J(\lambda_2, \alpha, \delta)$, then there exists two functions $h_1, h_2 \in P_k(\delta)$, such that, from Definition 1.1 and Theorem 2.1

$$(l-\lambda_2)\left[\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\right] + \lambda_2\,[z\alpha f''(z) + f'(z)] = h_1(z)$$

and

$$\alpha f'(z) + (1-\alpha)\frac{f(z)}{z} = h_2(z)$$

Hence

$$(1-\lambda_1)\left[\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\right] + \lambda_1\,[z\alpha f''(z) + f'(z)]$$

$$= \frac{\lambda_1}{\lambda_2} h_1(z) + \left(1 - \frac{\lambda_1}{\lambda_2}\right) h_2(z) \qquad (2.3)$$

Since the class $P_k(\delta)$ is a convex set, [2], it follows that the right hand side of (2.3) belongs to $P_k(\delta)$ and this proves the result.

Theorem 2.4 Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in J(\lambda, \alpha, \delta)$$

Then

$$|a_n| \leq \frac{k(1-\delta)}{\lambda + \alpha}$$

The function $f_{\lambda,\alpha,\delta}(z)$ defined as

$$\left(\frac{f_{\lambda,\alpha,\delta}(z)}{z}\right) = \frac{\alpha}{\lambda}\int_0^1 \left[\left(\frac{k}{4} + \frac{1}{2}\right) t^{\frac{\alpha}{\lambda}-1}\,\frac{1-(1-2\delta)\,tz}{1+tz}\right] dt$$

shows that this result is sharp.

Proof. Since $f \in J(\lambda, \alpha, \delta)$

$$\left\{(1-\lambda)\left[\alpha f'(z) + (1-\alpha)\frac{f(z)}{z}\right] + \lambda\,[z\,\alpha f''(z) + f'(z)]\right\}$$

$$= 1 + \sum_{n=1}^{\infty} c_n z^n$$

$$\in P_k(\delta)$$

It is known that $|c_n| \leq k(1-\delta)$ for all n. Using the above inequality, we prove the required result.

Theorem 2.5 (Covering theorem), Let $\lambda > 0$ and $0 < \lambda < 1$. Let $f = F \in J(\lambda, l, \delta)$. If D is the boundary of the image of U under F, then every point of D has a distance of atleast $\dfrac{\lambda+1}{(2+k) + 2\lambda - k\delta}$ from the origin.

Proof. Let $F(z) \neq \omega_0$ and $\omega_0 \neq 0$.

Then $f_1(z) = \dfrac{\omega_0\, F(z)}{\omega_0 + F(z)}$ is univalent in U. Since F is univalent.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then $f_1(z) = z + \left(a_2 - \frac{1}{\omega_0} \right) z^2 + \dots$

But $\left| a_2 - \frac{1}{\omega_0} \right| \leq 2$

By Theorem 2.4 $|a_2| \leq \frac{k(1-\delta)}{1+\lambda}$

Hence we obtain $|\omega_0| \geq \frac{1+\lambda}{(2+k)+2\lambda - k\lambda}$

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